HEATH-JARROW-MORTON FRAMEWORK

VLADIMIR PITERBARG

1. Setup

We will assume that all our **random variables**[1] and **stochastic processes** live on a **probability space** $(\Omega, \mathcal{F}, \mathbf{P})$ with a **sample space** Ω, **sigma-algebra** $\mathcal{F}$ and **probability measure** $\mathbf{P}$. We also assume that there exists a **filtration**

$$\{\mathcal{F}_t, \quad 0 \le t < \infty\}$$

such that

$$\mathcal{F}_s \subset \mathcal{F}_t \subset \mathcal{F}$$

for

$$0 \le s \le t < \infty.$$

We will be considering only those stochastic processes that have continuous trajectories $\mathbf{P}$-almost surely. In addition we will only be dealing with $\mathcal{F}_t$-**adapted processes**. Note that a process with continuous trajectories is **predictable** (or, as some call it, **previsible**) if it is **adapted**.

2. Paradigm

The concept of a replicating strategy is an important one. Let us recall its definition. Suppose we have an economy with two primary instruments. Their prices at time t will be denoted by I_t and J_t. A strategy is a pair of (predictable) processes (ϕ_t, ψ_t), where we interpret ϕ_t as the amount of instrument I we hold at time t, and ψ_t as the amount of instrument J we hold at time t. Then the value of the portfolio at time t is equal to

$$\Pi_t = \phi_t I_t + \psi_t J_t.$$

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[1]Words in **bold** denote concepts that we use extensively but whose definitions should be looked up elsewhere

The strategy is called *self-financing* if the change in the value of the portfolio comes from changes in the prices of the instruments only,

$$d\Pi_t = \phi_t \, dI_t + \psi_t \, dJ_t.$$

It is called *replicating* for a claim with a (random) payoff X at time T if

$$\Pi_T = X$$

$\mathbf{P}$-almost surely.

We will price derivative securities by constructing replicating strategies. We will use the fact that if the value of a replicating strategy (by definition self-financing) at option's maturity is equal to the payoff of the option, then the no-arbitrage value of the option today is equal to the initial investment required to put the replicating strategy in place. The fact that we can price derivative securities this way follows from the No Free Lunch condition

NFL: If two strategies have the same payoffs in the future, they must have the same value today.

From now on, we denote the value at time t of the instrument that pays X at some future time t by

$$\pi_t(X).$$

In the discussion above, it follows from NFL that $\pi_t(X)$ at time t *must* be equal to Π_t.

If we have two traded instruments, how can we replicate one with the other? The following theorem provides a partial answer, see [KS, p.182].

Theorem 2.1 (Martingale Representation Theorem)**.** *Let the filtration $\{\mathcal{F}_t\}$ be generated by a process W_t which is a* ***Brownian motion*** *under some measure $\mathbf{Q}$ on $(\Omega, \mathcal{F})$. Let the process N_t be an $\mathcal{F}_t$-adapted continuous square-integrable* ***martingale*** *under the same measure. Then there exists a predictable process v_t such that*

$$\mathbf{E}^Q \int_0^T v_t^2 \, dt < \infty$$

and

$$N_t = N_0 + \int_0^t v_s \, dW_s.$$

Moreover, if

$$v_s > 0 \qquad \mathbf{Q}\text{-}a.s.$$

and M_t *is another* $\mathcal{F}_t$*-adapted continuous square-integrable martingale (under* $\mathbf{Q}$*) then there exists another predictable process* ϕ_t *such that*

$$M_t = M_0 + \int_0^t \phi_s \, dN_s.$$

We can interpret the theorem as that we can "replicate" one martingale M by "holding" the amount of ϕ_t of martingale N at time t. We will make use of this theorem by finding a measure under which the values of traded instruments are martingales, and then replicating them using the theorem.

3. Interest rate market

The price at time t of a zero bond that pays \$1 at time T is denoted by

$$P(t, T).$$

The instantaneous forward rate at time t for forward period $[T, T+0]$ is denoted by

$$f(t, T).$$

The short rate at time t is denoted by

$$r(t) = f(t, t).$$

Recall that by definition

$$P(t, T) = \exp\left(-\int_t^T f(t, u)\, du\right).$$

We can go back and forth between bond prices and instantaneous forward rates. If we impose some evolution on forward rates, that immediately "locks in" the evolution of bond prices, and vice versa.

Note that for bonds $P(t, T)$ and forward rates $f(t, T)$ the second argument (capital T) works as an "index" that differentiates various traded instruments. The first index (little t) is a time index. For example, if T is fixed and t changes, we get a stochastic process of bond values $\{P(t, T)\}_{t=0}^T$ (generally we assume that little t cannot go beyond capital T as it would not make much economic sense). We assume that for each T, this process is $\mathcal{F}_t$-adapted and continuous. Then the same for the stochastic process $\{f(t, T)\}_{t=0}^T$.

4. What is no-arbitrage?

Suppose we try to create an interest-rate model by imposing some dynamics on random processes for bond process $P(t,T)$. Obviously we would have to do that for each maturity T. How do we decide whether our model makes sense? Consider for simplicity one-factor model (only one source of noise). We will show later that in this situation a bond with some maturity S can be dynamically replicated using another bond with maturity T as long as $T > S$. So a bond with maturity S will have two prices, one given by our original model specification (that at time $t = 0$ is equal to the market price of a bond $P(0,S)$), and the other price obtained from replicating strategy.

> The only way for the model to make sense (be a "no-arbitrage" model) is to have those two prices coincide.

Heath-Jarrow-Morton approach allows us to construct models that prevent this kind of "model arbitrage", where the same instrument can have two different prices depending on what angle you look at them from.

5. Single-factor HJM

5.1. **Assumptions.** Let us start by fixing a Brownian motion W_s under the (original) measure $\mathbf{P}$, and by assuming that the filtration $\{\mathcal{F}_t\}$ is generated by W. We further assume a very general form for forward rates,

(5.1)
$$f(t,T) = f(0,T) + \int_0^t \sigma(s,T)\, dW_s + \int_0^t \alpha(s,T)\, ds, \qquad 0 \le t \le T.$$

Here $\sigma(s,T)$ and $\alpha(s,T)$ are very general processes, that are assumed to be (some technical conditions)

- adapted and continuous processes of s;
- square-integrable in s;
- square-integrable in both arguments.

Specifying forward rates' evolution by (5.1) completely fixes up the model. We derive the processes for other quantities below.

5.2. **Bond prices.** We have

(5.2)

$$\begin{aligned} P(t,T) &= \exp\left(-\int_t^T f(t,u)\,du\right) \\ &= \exp\left(-\int_t^T \left[f(0,u)+\int_0^t \sigma(s,u)\,dW_s+\int_0^t \alpha(s,u)\,ds\right]du\right) \\ &= \exp\left(-\int_0^t \left(\int_t^T \sigma(s,u)\,du\right)dW_s \right. \\ &\quad \left. -\int_t^T f(0,u)\,du-\int_0^t\int_t^T \alpha(s,u)\,du\,ds\right). \end{aligned}$$

5.3. **Money market.** A money market B_t is defined as follows

$$dB_t = r(t)\,B_t\,dt, \quad B_0 = 1,$$

which can also be written as

$$B_t = \exp\left(\int_0^t r(s)\,ds\right).$$

Money market at time t is the value of one dollar invested in time zero and rolled up every day at then-prevailing short rate.

Recall that

$$r(t) = f(t,t),$$

so that (from (5.1))

$$r(u) = f(0,u)+\int_0^u \sigma(s,u)\,dW_s+\int_0^u \alpha(s,u)\,ds.$$

Then

(5.3)

$$\begin{aligned} B_t &= \exp\left(\int_0^t \left[f(0,u)+\int_0^u \sigma(s,u)\,dW_s+\int_0^u \alpha(s,u)\,ds\right]du\right) \\ &= \exp\left(\int_0^t \left(\int_s^t \sigma(s,u)\,du\right)dW_s+\int_0^t f(0,u)\,du+\int_0^t\int_s^t \alpha(s,u)\,du\,ds\right) \end{aligned}$$

5.4. **Discounted bonds.** Let us denote

$$Z(t,T) = B_t^{-1}P(t,T).$$

What we have here is that $Z(t,T)$ is the value in today's ($t=0$) dollars of a bond at time t. The reason to consider discounted values is that we would like to be able to compare values of instruments at different times, and they are directly compatible only if expressed in the same units, in this case being "time 0 dollars". Combining (5.2) and (5.3) we get

$$
\begin{aligned}
Z(t,T) &= B_t^{-1}P(t,T) \\
&= \exp\left(-\int_0^t \left(\int_s^t \sigma(s,u)\,du\right) dW_s\right. \\
&\quad \left. -\int_0^t f(0,u)\,du - \int_0^t \int_s^t \alpha(s,u)\,du\,ds\right) \\
&\quad \times \exp\left(-\int_0^t \left(\int_t^T \sigma(s,u)\,du\right)\,dW_s\right. \\
&\quad \left. -\int_t^T f(0,u)\,du - \int_0^t \int_t^T \alpha(s,u)\,du\,ds\right) \\
&= \exp\left(-\int_0^t \left(\int_s^t \sigma(s,u)\,du + \int_t^T \sigma(s,u)\,du\right) dW_s\right) \\
&\quad \times \exp\left(-\int_0^t f(0,u)\,du - \int_t^T f(0,u)\,du\right) \\
&\quad \times \exp\left(-\int_0^t \int_s^t \alpha(s,u)\,du\,ds - \int_0^t \int_t^T \alpha(s,u)\,du\,ds\right) \\
&= \exp\left(-\int_0^t \left(\int_s^T \sigma(s,u)\,du\right) dW_s\right) \\
&\quad \times \exp\left(-\int_0^T f(0,u)\,du\right) \\
&\quad \times \exp\left(-\int_0^t \int_s^T \alpha(s,u)\,du\,ds\right) \\
&= \exp\left(\int_0^t \Sigma(s,T)\,dW_s\right. \\
&\quad \left. -\int_0^T f(0,u)\,du - \int_0^t \int_s^T \alpha(s,u)\,du\,ds\right),
\end{aligned}
$$

where

$$\Sigma(s,T) = -\int_s^T \sigma(s,u)\,du.$$

In particular (Ito's lemma)

$$dZ(t,T) = Z(t,T)\left(\Sigma(t,T)\,dW_t - \left(\int_s^T \alpha(t,u)\,du\right)dt\right) + Z(t,T)\left(\frac{1}{2}\Sigma^2(t,T)\,dt\right). \tag{5.4}$$

5.5. **Replicating an arbitrary payoff: the steps.** As was mentioned before, we would like to construct a strategy for replicating an S-maturity bond with a strategy involving a T-maturity bond. In fact, we can do one better. We can show that we can replicate *any payoff* using only T-maturity bond (and money-market account). Assume that X is a payoff that we want to replicate. We assume that it is an $\mathcal{F}_S$-measurable random variable where $S < T$ (if you recall the definition of an $\mathcal{F}_S$-measurable random variable, it just means that the value X will be "known" come time S).

Recall a theorem presented earlier about replicating one martingale with another. If we could make all relevant processes martingales, we would use that theorem to construct a replicating strategy. So our plan is as follows,

1. Find a measure $\mathbf{Q}$ under which $Z(t,T)$ is a martingale;
2. Find a $\mathbf{Q}$-martingale V_t such that at the final time S the martingale V is equal to the discounted value of the payoff, $V_S = B_S^{-1}X$;
3. Find a predicted process ϕ_t such that $V_t = V_0 + \int_0^t \phi_t\, dZ(t,T)$;
4. Prove that ϕ_t is (part of) a replicating strategy.

Let us do these steps.

5.5.1. *Step 1.* Rearranging (5.4) we get,

$$\begin{aligned} dZ(t,T) &= Z(t,T)\left(\Sigma(t,T)\,dW_t + \left(\frac{1}{2}\Sigma^2(t,T) - \int_s^T \alpha(t,u)\,du\right)dt\right) \\ &= Z(t,T)\,\Sigma(t,T)\left(dW_t + \left(\frac{1}{2}\Sigma(t,T) - \frac{1}{\Sigma(t,T)}\int_t^T \alpha(t,u)\,du\right)dt\right). \end{aligned} \tag{5.5}$$

Changes of measure are the same as changes of drift for the Brownian motion (Girsanov's theorem). The process $Z(t,T)$ will be a martingale if an SDE for Z does not have the "dt" part. So if we define

$$\begin{aligned} \gamma_t &= \frac{1}{2}\Sigma(t,T) - \frac{1}{\Sigma(t,T)}\int_t^T \alpha(t,u)\,du, \\ d\tilde{W}_t &= dW_t + \gamma_t\,dt, \end{aligned} \tag{5.6}$$

and define measure $\mathbf{Q}$ by the requirement that $\tilde{W}_t$ be a Brownian motion under $\mathbf{Q}$, we would have

$$dZ(t,T) = Z(t,T)\,\Sigma(t,T)\,d\tilde{W}_t,$$

which defines a martingale.

5.5.2. *Step 2.* The process

$$V_t = \mathbf{E}^Q\left(B_S^{-1}X\middle|\mathcal{F}_t\right)$$

is automatically a martingale because of a tower rule. If $s < t < S$ then

$$\begin{aligned}\mathbf{E}^Q\left(V_t\middle|\mathcal{F}_s\right) &= \mathbf{E}^Q\left(\mathbf{E}^Q\left(B_S^{-1}X\middle|\mathcal{F}_t\right)\middle|\mathcal{F}_s\right)\\ &= \mathbf{E}^Q\left(B_S^{-1}X\middle|\mathcal{F}_s\right)\\ &= V_s.\end{aligned}$$

Clearly

$$V_S = \mathbf{E}^Q\left(B_S^{-1}X\middle|\mathcal{F}_S\right).$$

Since X is $\mathcal{F}_S$-measurable, and so is B_S^{-1}, we have from one of the properties of conditional expectation that

$$V_S = \mathbf{E}^Q\left(B_S^{-1}X\middle|\mathcal{F}_S\right) = B_S^{-1}X$$

as required.

5.5.3. *Step 3.* By Martingale Representation Theorem there exists a predictable process ϕ_t such that

$$dV_t = \phi_t\,dZ(t,T),$$

so that

$$\begin{aligned}V_t &= V_0 + \int_0^t \phi_s\,dZ(s,T)\\ &= \mathbf{E}^Q\left(B_S^{-1}X\middle|\mathcal{F}_0\right) + \int_0^t \phi_s\,dZ(s,T).\end{aligned}$$

5.5.4. *Step 4.* Define

$$\psi_t = V_t - \phi_t Z(t,T).$$

Then the strategy requires holding at time t

- ϕ_t units of bond $P(t,T)$; and
- ψ_t units of the money market account B_t.

At time t the value of the portfolio R_t is equal to

$$
\begin{aligned}
R_t &= \phi_t P(t,T) + \psi_t B_t \\
&= \phi_t P(t,T) + (V_t - \phi_t Z(t,T)) B_t \\
&= \phi_t P(t,T) + \left(V_t - \phi_t B_t^{-1} P(t,T)\right) B_t \\
&= \phi_t P(t,T) + V_t B_t - \phi_t P(t,T) \\
&= V_t B_t.
\end{aligned} \tag{5.7}
$$

In particular, at time S, $V_S = B_S^{-1} X$ (see Step 3) so that

$$R_S = V_S B_S = B_S B_S^{-1} X = X.$$

At the expiry time S we exactly replicate the payoff X.

We need to check if the strategy is self-financing. We have (B_t has no stochastic term, we use that a few times below)

$$
\begin{aligned}
dR_t &= B_t\, dV_t + V_t\, dB_t \\
&= B_t\, \phi_t\, dZ(t,T) + V_t\, dB_t \\
&= [\phi_t\, d(B_t Z(t,T)) - \phi_t Z(t,T)\; dB_t] + V_t\, dB_t \\
&= \phi_t\, dP(t,T) + (V_t - \phi_t Z(t,T))\; dB_t \\
&= \phi_t\, dP(t,T) + (V_t - \phi_t Z(t,T))\; dB_t \\
&= \phi_t\, dP(t,T) + \psi_t\, dB_t.
\end{aligned} \tag{5.8}
$$

Therefore, it is indeed self-financing.

5.6. **Replicating an arbitrary payoff: results.** Let us recap. We have constructed a self-financing (see (5.8)), replicating (see (5.7)) portfolio consisting of a T-maturity bond $P(t,T)$ and a market money account B_t for an arbitrary payoff X. Therefore, the value of that payoff today ($t = 0$) is just the value of this strategy today, so we have

$$\pi_0(X) = R_0 = V_0 B_0 = V_0 = \mathbf{E}^Q \left(B_S^{-1} X \middle| \mathcal{F}_0 \right) = \mathbf{E}^Q \left(B_S^{-1} X \right).$$

Likewise, the value of it at some intermediate time t is equal to

$$\pi_t(X) = R_t = V_t B_t = B_t \mathbf{E}^Q \left(B_S^{-1} X \middle| \mathcal{F}_t \right). \tag{5.9}$$

The importance of these formulas cannot be overemphasized. They show that the value of any derivative only depends on the dynamics of bonds in the measure $\mathbf{Q}$, and not the original measure $\mathbf{P}$. Measure $\mathbf{Q}$ has a special name, a *risk-neutral measure*.

5.7. **Replicating one bond with another.** So let us go back to the problem of replicating a bond $P(t,S)$ using $P(t,T)$ and money market B_t. The bond $P(t,S)$ is just a "derivative security" with a (constant) payoff of \$1 at time S, $X \equiv 1$ paid at S. By (5.9) we must have

$$P(t,S) = B_t \mathbf{E}^Q \left(B_S^{-1} \middle| \mathcal{F}_t \right). \tag{5.10}$$

Slight rewrite yields

$$B_t^{-1} P(t,S) = \mathbf{E}^Q \left(B_S^{-1} P(S,S) \middle| \mathcal{F}_t \right)$$

so (5.10) is equivalent to the requirement that

$$Z(t,S) = B_t^{-1} P(t,S)$$

be a martingale under the same measure $\mathbf{Q}$!

Of course we have from (5.5) and (5.6)

$$\begin{aligned} dZ(t,S) &= Z(t,S)\Sigma(t,S)\left[dW_t + \left(\frac{1}{2}\Sigma(t,S) - \frac{1}{\Sigma(t,S)} \int_t^S \alpha(t,u)\,du \right) dt \right] \\ &= Z(t,S)\Sigma(t,S)\left[d\tilde{W}_t - \left(\frac{1}{2}\Sigma(t,T) - \frac{1}{\Sigma(t,T)} \int_t^T \alpha(t,u)\,du \right) dt \right. \\ &\qquad \left. + \left(\frac{1}{2}\Sigma(t,S) - \frac{1}{\Sigma(t,S)} \int_t^S \alpha(t,u)\,du \right) dt \right]. \end{aligned}$$

The only way for $Z(t,S)$ to be a martingale is to have the "dt" term absent. That requires

$$\frac{1}{2}\Sigma(t,T) - \frac{1}{\Sigma(t,T)} \int_t^T \alpha(t,u)\,du = \frac{1}{2}\Sigma(t,S) - \frac{1}{\Sigma(t,S)} \int_t^S \alpha(t,u)\,du.$$

This should hold for all T and S. Therefore, we must have (γ_t is independent of T)

$$\frac{1}{2}\Sigma(t,T) - \frac{1}{\Sigma(t,T)} \int_t^T \alpha(t,u)\,du = \gamma_t,$$

so that

$$\int_t^T \alpha(t,u)\,du = \frac{1}{2}\Sigma^2(t,T) - \Sigma(t,T)\gamma_t.$$

Differentiating with respect to T we get

$$\alpha(t,T) = \Sigma(t,T)\frac{\partial \Sigma(t,T)}{\partial T} - \gamma_t \frac{\partial \Sigma(t,T)}{\partial T}.$$

Since

$$\frac{\partial \Sigma(t,T)}{\partial T} = -\sigma(t,T),$$

we finally have

$$\begin{aligned}(5.11)\qquad \alpha(t,T) &= -\Sigma(t,T)\,\sigma(t,T)+\gamma_t\sigma(t,T)\\ &= \sigma(t,T)\left(\gamma_t-\Sigma(t,T)\right).\end{aligned}$$

Recall that for a Black-Scholes model, the dynamics of stock under measure $\mathbf{P}$ was arbitrary. One could use a geometric Brownian motion with an arbitrary drift, or almost any other Ito process. Here we have quite a different situation – for a model to be no-arbitrage, some restrictions on drift (in the real-world measure $\mathbf{P}$!) have to be imposed. Otherwise, *arbitrage between bonds is possible*. Yuri will show that his statistical model does not satisfy (5.11) and then modify it slightly so that the condition (5.11) is fullfilled.

6. Model properties under risk-neutral measure

We have shown that for any model that satisfies (5.11) we can construct a measure $\mathbf{Q}$ with a rather remarkable property: for any traded asset A_t its discounted value is a martingale under measure $\mathbf{Q}$,

$$B_t^{-1}A_t \triangleq e^{-\int_0^t r(s)\,ds}A_t \qquad \text{is a } \mathbf{Q}\text{-martingale.}$$

This of course holds true for all bonds as well. Now we can forget about the original measure $\mathbf{P}$, and value all instruments by taking expected values (that are just integrals really) under measure $\mathbf{Q}$ just like in (5.9).

Let us use this very important fact to derive the equations for bond prices and forward rates. We have under $\mathbf{Q}$,

$$dZ(t,T) = Z(t,T)\,\Sigma(t,T)\;d\tilde{W}_t.$$

Also

$$\begin{aligned} dP(t,T) &= d\left(B_tZ(t,T)\right)\\ &= Z(t,T)\,dB_t + B_t\,dZ(t,T)\\ &= r(t)\,Z(t,T)\,B_t\,dt + B_tZ(t,T)\,\Sigma(t,T)\;d\tilde{W}_t\\ &= r(t)\,P(t,T)\;dt + P(t,T)\,\Sigma(t,T)\;d\tilde{W}_t.\end{aligned}$$

So all bonds (in fact, all traded instruments) have the same rate of return $r(t)$ under $\mathbf{Q}$ (this is why it is called risk-neutral measure).

Recall that we have for the forward rates

$$f(t,T) = -\frac{\partial}{\partial T}\log P(t,T),$$

and (under $\mathbf{Q}$)

$$P(t,T) = P(0,T)\exp\left(\int_0^t\left[r(s)-\frac{1}{2}\Sigma^2(s,T)\right]ds + \int_0^t \Sigma(s,T)\;d\tilde{W}_s\right).$$

Thus

$$\log P(t,T) = \log P(0,T) + \int_0^t \left[r(s) - \frac{1}{2}\Sigma^2(s,T) \right] ds + \int_0^t \Sigma(s,T)\, d\tilde{W}_s$$

and

$$\begin{aligned} \frac{\partial}{\partial T} \log P(t,T) &= \frac{\partial}{\partial T} \log P(0,T) \\ &\quad + \frac{\partial}{\partial T} \left(\int_0^t \left[r(s) - \frac{1}{2}\Sigma^2(s,T) \right] ds + \int_0^t \Sigma(s,T)\, d\tilde{W}_s \right) \\ &= \frac{\partial}{\partial T} \log P(0,T) \\ &\quad + \int_0^t \left[-\frac{1}{2} \left(\frac{\partial}{\partial T} \Sigma^2(s,T) \right) \right] ds + \int_0^t \left(\frac{\partial}{\partial T} \Sigma(s,T) \right) d\tilde{W}_s \end{aligned}$$

Since

$$\frac{\partial}{\partial T} \Sigma(s,T) = -\sigma(s,T)$$

we get

$$\frac{\partial}{\partial T} \log P(t,T) = \frac{\partial}{\partial T} \log P(0,T) + \int_0^t \Sigma(s,T)\,\sigma(s,T)\, ds - \int_0^t \sigma(s,T)\, d\tilde{W}_s.$$

Hence

$$\begin{aligned} f(t,T) &= -\frac{\partial}{\partial T} \log P(t,T) \\ &= f(0,T) - \int_0^t \Sigma(s,T)\,\sigma(s,T)\, ds + \int_0^t \sigma(s,T)\, d\tilde{W}_s. \end{aligned}$$

In differential form

$$df(t,T) = -\Sigma(t,T)\,\sigma(t,T)\, dt + \sigma(t,T)\, d\tilde{W}_t. \tag{6.1}$$

Note that the drift of forward rates (under $\mathbf{Q}$) is equal to

$$-\Sigma(t,T)\,\sigma(t,T) = \sigma(t,T) \int_0^T \sigma(t,\tau)\, d\tau \tag{6.2}$$

and is completely determined by volatility $\sigma(\cdot,\cdot)$. *Any model that satisfies (6.1) and (6.2) is called a one-factor HJM model.*

7. Multi-factor HJM

7.1. **Motivation.** In the model we just constructed, all forward rates are instantaneously perfectly correlated. This is not very realistic – a move in 30y forward rate cannot usually be completely predicted from a move in an overnight rate. While one-factor models have their place in the arsenal of a Quant, *multi-factor* models are needed for pricing more complex instruments. For example, if you consider an option on a spread between 3m Libor rate and 10y swap rate, a one-factor model will severely underprice it. By "multi-factor" models we understand models that have more than one Brownian motion as stochastic drivers.

7.2. **Construction.** To construct a multi-factor HJM we can jump through all the same hoops as we just did for a one-factor HJM. Practitioners however, rarely bother to do so more than once in their lifetime. They use a shortcut which we now proceed to share with you.

Recall the *Fundamental Theorem of Arbitrage Pricing* (see [SC]),

> Absence of arbitrage is equivalent to the existence of a risk-neutral measure. Under risk-neutral measure, all traded assets must have the same rate of return.

Recall that the money-market account is governed by the equation

$$dB_t = r\left(t\right) B_t \, dt.$$

This equation is valid under any measure equivalent to $\mathbf{P}$, because there are now stochastic terms. Hence, a money-market account has a rate of return $r\left(\cdot\right)$ under the risk-neutral measure (call it $\mathbf{Q}$ again), whose existence is guaranteed by the Fundamental Theorem.

Hence, under $\mathbf{Q}$, all bonds must satisfy

$$dP\left(t,T\right) = r\left(t\right) P\left(t,T\right) \, dt + P\left(t,T\right) \left[\Sigma_1\left(t,T\right) \, d\tilde{W}_t^1 + \cdots + \Sigma_N\left(t,T\right) \, d\tilde{W}_t^N\right],$$

where $\left(\tilde{W}_t^1, \ldots, \tilde{W}_t^N\right)$ is an N-dimensional Brownian motion (with independent components). Solving the equation we get

$$\begin{aligned} P\left(t,T\right) &= P\left(0,T\right) \exp\left(\int_0^t \left[r\left(s\right) - \frac{1}{2}\left(\Sigma_1^2\left(s,T\right) + \cdots + \Sigma_N^2\left(s,T\right)\right)\right] ds \right. \\ &\quad \left. + \int_0^t \Sigma_1\left(t,T\right) \, d\tilde{W}_t^1 + \cdots + \int_0^t \Sigma_N\left(t,T\right) \, d\tilde{W}_t^N\right). \end{aligned}$$

Thus we have for forward rates (repeating previous arguments)

$$\begin{aligned}\frac{\partial}{\partial T}\log P(t,T) &= \frac{\partial}{\partial T}\log P(0,T) \\ &+ \int_0^t \left[-\frac{1}{2}\frac{\partial}{\partial T}\left(\Sigma_1^2(s,T)+\cdots+\Sigma_N^2(s,T)\right)\right] ds \\ &+ \int_0^t \left(\frac{\partial}{\partial T}\Sigma_1(t,T)\right) d\tilde{W}_t^1 + \cdots + \int_0^t \left(\frac{\partial}{\partial T}\Sigma_N(t,T)\right) d\tilde{W}_t^N\end{aligned}$$

Define

$$\sigma_n(s,T) = -\frac{\partial}{\partial T}\Sigma_n(s,T), \quad n = 1,\ldots,N. \tag{7.1}$$

Then

(7.2)

$$\begin{aligned}f(t,T) &= -\frac{\partial}{\partial T}\log P(t,T) \\ &= f(0,T) - \int_0^t \left[\Sigma_1(s,T)\sigma_1(s,T)+\cdots+\Sigma_N(s,T)\sigma_N(s,T)\right] ds \\ &+ \int_0^t \sigma_1(s,T)\, d\tilde{W}_t^1 + \cdots + \int_0^t \sigma_1(s,T)\, d\tilde{W}_t^N.\end{aligned}$$

This is it! A multi-factor HJM model is any model that satisfies (7.2) and (7.1). It is automatically a no-arbitrage model by the Fundamental Theorem. We took much less time (and space) to construct it using the shortcut.

Naturally, there was some value in starting in real-world measure **P** and then making our way into risk-neutral measure **Q**. The circuitous way we undertook in the early chapters for one-factor model gave us

- A constructive way of finding the risk-neutral measure;
- A connection between real and risk-neutral worlds;
- An understanding of where it came from; and
- An intimate connection between replication and pricing.

But if you have done it once, there is rarely a reason to do it for the second time.

7.3. **Pricing.** Well, we are not quite finished with the multi-factor model yet. We still need a formula for valuing an arbitrary traded asset.

If A_t is the price at time t of some traded asset t, it must satisfy the SDE

$$dA_t = r(t)\, A_t\, dt + \text{stochastic terms},$$

where

$$\text{stochastic terms} = a_1(t)\, d\tilde{W}_t^1 + \cdots + a_N(t)\, d\tilde{W}_t^N$$

for some stochastic processes $\{a_n(t)\}$. In particular, the discounted value $B_t^{-1}A_t$ satisfies

$$\begin{aligned} d\left(B_t^{-1}A_t\right) &= A_t\, dB_t^{-1} + B_t^{-1}\, dA_t \\ &= -r(t)\, A_t B_t^{-1}\, dt + B_t^{-1}\left[r(t)\, A_t\, dt + \text{stochastic terms}\right] \\ &= B_t^{-1} \times [\text{stochastic terms}] \end{aligned}$$

so that

$$B_t^{-1}A_t \text{ is a } \mathbf{Q}\text{-martingale.}$$

In particular, if

$$A_S = X$$

is an $\mathcal{F}_S$-measurable payoff, then

$$\begin{aligned} B_t^{-1}\pi_t(X) &= B_t^{-1}A_t \\ &= \mathbf{E}^Q\left(B_S^{-1}A_S \middle| \mathcal{F}_t\right) \\ &= \mathbf{E}^Q\left(B_S^{-1}X \middle| \mathcal{F}_t\right), \end{aligned}$$

so that

$$\pi_t(X) = B_t\mathbf{E}^Q\left(B_S^{-1}X \middle| \mathcal{F}_t\right) \tag{7.3}$$

Not surprisingly, the valuation formula (7.3) is the same as (5.9) for one-factor case.

In fact, specifying some measure $\mathbf{Q}$, a law for the "numeraire", the money market process B_t, and the formula (7.3) is (almost) all that is really needed to construct any no-arbitrage interest-rate model. This approach covers a wider range of models than HJM framework does. In particular, models with jumps are also included, as are those in which instanteneous forward rates do not exist. It is a different story alltogether, but if you are interested, start with [FH] and [KH].

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